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Strength Margins for Combined Random Stresses

J. R. Fuller*
The Boeing Company, Seattle, Wash.

Statistical mechanics procedures are widely used in the aerospace industry to analyze the effects of random loading on flight vehicles. Although existing procedures give the structures analyst considerable knowledge of the loads to which various parts of the vehicle are subjected, they do not provide a usable procedure for evaluating the structural strength of an element for combined random stresses. A procedure is derived herein, which permits a determination of the number of times per unit time that various strength margin levels are exceeded for combined random stresses.

Nomenclature

x, ξ = shear stress, psi
y, f = axial stress, psi
x_0, ξ_0, y_0, f_0 = steady stress values, shear and axial stress, respectively, psi
F_ξ = allowable shear stress, psi
F_t, F_c = allowable axial stress, psi
$\mathbf{z}$ = stress vector, $x\mathbf{i} + y\mathbf{j}$
$\dot{\mathbf{z}}$ = stress velocity vector, $\alpha\mathbf{i} + \beta\mathbf{j}$
$\sigma_x, \sigma_y, \sigma_\alpha, \sigma_\beta$ = root mean square values (standard deviations) for x, y, α, and β, respectively
$\Phi_x(\omega), \Phi_y(\omega)$ = power spectral density functions for random process $x(t)$ and $y(t)$, respectively
$\Phi_{xy}(\omega)$ = cross power spectral density function for $x(t)$ and $y(t)$
ω = circular frequency, rad/sec
ρ = correlation coefficient for $x(t)$ and $y(t)$
α, β = time rate of change of x and y, respectively
$p(x), p(y)$ = probability densities of x and y
$p(x, y)$ = joint probability density of x and y
$f(x, \alpha, y, \beta)$ = probability density of x, α, y, β
MS = margin of safety
$P(MS < 0)$ = probability that MS is less than zero, or percent time that MS is less than zero
$P(MS \geqslant 0)$ = probability that MS is greater than or equal to zero
C = a curve on the xy plane
$\mathbf{N}$ = unit vector normal on C
[ ] = column matrix
$\lfloor\ \rfloor$ = row matrix
N_c = crossings of an arbitrary curve/sec or /ft traveled
σ_w = root mean square gust velocity, fps
$f(\sigma_w)$ = probability density distribution of σ_w
V = velocity, fps
$\bar{G}$ = expected exceedances of limit design strength/hr
$\bar{A}$ = root mean square stress response for σ_w of unity, psi/fps
N_0 = number of times/unit time or distance that a time-history crosses its mean value with positive or negative slope

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* Supervisor, Product Development, Structures Research Group, Structures Staff.

Introduction

THE strengths of various structural elements in aircraft and aerospace vehicles are usually defined in terms of a stress interaction function or interaction diagram such as that shown in Fig. 1. This type of diagram shows the various combinations of stresses which would cause the structural element to fail. Any combination of stresses within the envelope is allowable.

When a flight vehicle is subjected to random loading, such as continuous atmospheric turbulence, buffeting, a turbulent boundary layer, or engine noise, random stress components are generated in the various structural elements. It is important for the structures engineer to know the expected number of years, hours, or minutes that the structural element can sustain the combined random stresses before the strength is exceeded.

Development of the Procedure

Joint Probability Approach

For purposes of illustration, let us consider two stress components, axial stress and shear stress, on a segment of a stiffened skin panel, as shown in Fig. 1. Further, let us assume that each stress time-history is statistically stationary and has a Gaussian probability distribution when sampled at equal increments of time. If the time-histories were statistically independent, that is, if there were no correlation between them, their joint probability density function would be the simple product of their individual probability densities.

$$p(x, y) = p(x)p(y) = [1/(2\hat{\pi}\sigma_x\sigma_y)] \exp\{-\tfrac{1}{2}[x^2/\sigma_x^2 + y^2/\sigma_y^2]\} \quad (1)$$

However, if the time-histories were correlated, the joint probability density would be as follows:

$$p(x, y) = 1/[2\pi\sigma_x\sigma_y(1 - \rho^2)^{1/2}] \exp(\{(-1)/[2(1 - \rho^2)]\} \times [x^2/\sigma_x^2 - (2\rho xy)/(\sigma_x\sigma_y) + y^2/\sigma_{xy}^2]) \quad (2)$$

The probability that the combined stresses would not exceed the strength envelope enclosing a region R is the probability that the margin of safety (MS) would be not less than zero.

$$P(MS \geqslant 0) = \int_R\int p(x, y)dxdy \quad (3a)$$

where, the margin of safety is

$$MS = 1 - \frac{\text{actual stress state}}{\text{allowable stress state}} \quad (3b)$$

A stress state is defined herein as stress components having a unique ratio of one to the other.

On the other hand, the probability of having less than zero margin of safety at any instant in time is

$$P(MS < 0) = 1 - P(MS \geqslant 0) \quad (4)$$

Perhaps a more meaningful statistical estimate would be the expected time to exceed the design strength envelope. This type of estimate involves determining the expected number of times per unit time that the combined stress vector $\mathbf{z}(t)$ perforates the strength envelope. There must be at least one peak value for every two perforations.

Consider a random time-history of shear stress $\xi(t) = x(t)$ and of axial stress $\xi(t) = y(t)$. Then, the stress vector $\mathbf{z}(t)$ is

$$\mathbf{z}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad (5)$$

A schematic diagram showing the locus of $\mathbf{z}(t)$ as a function of time is shown in Fig. 2.

Let us assume, in the same manner as Bendat,[1] that the stress vector $\mathbf{z}(t)$ and its time derivative $\dot{\mathbf{z}}(t)$ are observed to follow a given joint probability density denoted as follows:

$$f(x, \alpha, y, \beta) = \text{Prob}[x < \xi \leq x + dx, \alpha < \dot{\xi} \leq \alpha + d\alpha, y < f \leq y + dy, \beta < \dot{f} \leq \beta + d\beta] \quad (6)$$

where

f = axial stress, psi
$\dot{f}$ = time rate of change of axial stress, psi/sec
ξ = shear stress, psi
$\dot{\xi}$ = time rate of change of shear stress, psi/sec

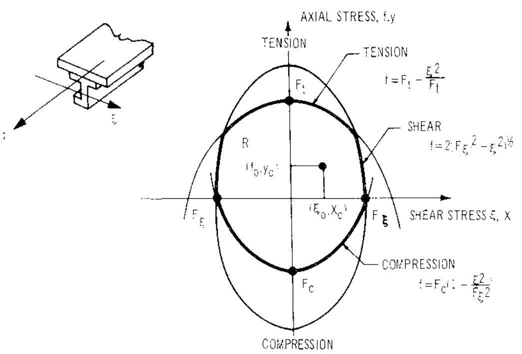

Fig. 1 Typical stress interaction diagram.

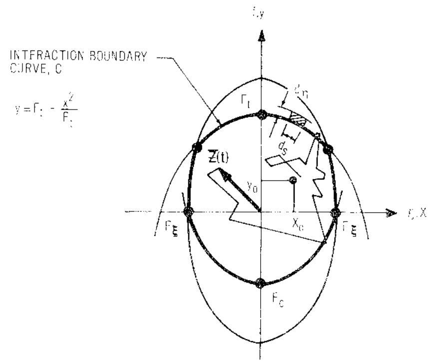

Fig. 2 Combined stress vector.

The preceding expression can be thought of as representing the fraction of time per unit time that the stress vector $\mathbf{z}(t)$ spends in the interval ($x, x + dx; y, y + dy$) with velocities in the interval ($\alpha, \alpha + d\alpha; \beta, \beta + d\beta$).

The expected number of times per unit time that $\mathbf{z}(t)$ will pass through the strength interaction boundary zone $d\eta$ (Fig. 2) is equal to the time spent in the interval $d\eta$ divided by the time required to cross the interval. The time required to cross the interval is equal to $d\eta$ divided by the velocity component of the stress vector normal to the interaction boundary curve, denoted as C.

A unit vector normal to C is

$$\mathbf{N} = \frac{\text{grad } C}{|\text{grad } C|} \quad (7)$$

If C is a function of x and y, such as

$$y = F_t - (x^2)/F_t \quad (8a)$$

where this function represents an interaction curve for shear and tension stresses, then

$$C = y - F_t + (x^2/F_t) = 0 \quad (8b)$$

The gradient of C is

$$\text{grad } C = \frac{\partial C}{\partial x}\mathbf{i} + \frac{\partial C}{\partial y}\mathbf{j} = \frac{2x}{F_t}\mathbf{i} + \mathbf{j} \quad (8c)$$

The modulus of this gradient vector is

$$\left[\left(\frac{\partial C}{\partial x}\right)^2 + \left(\frac{\partial C}{\partial y}\right)^2\right]^{1/2} = \frac{1}{F_t}[4x^2 + F_t^2]^{1/2} \quad (8d)$$

Therefore, the unit normal vector on C is

$$\mathbf{N} = \left[\frac{2x}{[4x^2 + F_t^2]^{1/2}}\right]\mathbf{i} + \left[\frac{F_t}{[4x^2 + F_t^2]^{1/2}}\right]\mathbf{j} \quad (8e)$$

The velocity vector $\dot{\mathbf{z}}(t)$ is

$$\dot{\mathbf{z}}(t) = \alpha(t)\mathbf{i} + \beta(t)\mathbf{j} \quad (8f)$$

The rate at which the interval $d\eta$ on C is traversed is the dot product of the unit normal vector and the velocity-vector.

$$\mathbf{N}\cdot\dot{\mathbf{z}}(t) = \frac{2\alpha x}{[4x^2 + F_t^2]^{1/2}} + \frac{\beta F_t}{[4x^2 + F_t^2]^{1/2}}$$

If the time required to cross the interval $d\eta$ is τ, then

$$\tau = \frac{d\eta}{|\text{grad } C/|\text{grad } C|\cdot\dot{\mathbf{z}}(t)|} \; (\tau > 0) \quad (9)$$

The expected number of times/unit time that the curve C is crossed with velocities ($\alpha, \alpha + d\alpha; \beta,\beta + d\beta$) denoted $N_{C_{\alpha\beta}}$ is equal to the time spent in the interval divided by the

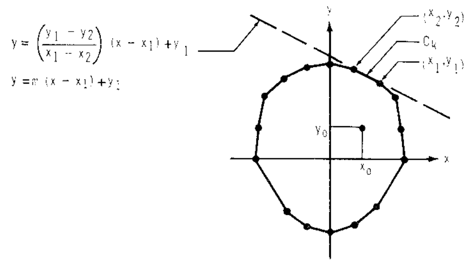

Fig. 3 Approximation to interaction diagram by straight-line segments.

time required to cross the interval.

$$N_{c_{\alpha\beta}} = \int_C \frac{\dfrac{[f(x, \alpha, y, \beta)d\alpha d\beta]d\eta}{d\eta}}{|\text{grad } C/|\text{grad } C|\cdot\dot{\mathbf{z}}(t)|}\, dS \tag{10}$$

The total number of passages, both out of the interval $d\eta$, from the region of positive margin of safety R, and into the interval, toward the region R, is obtained by integrating with respect to α and β for all possible velocities. Therefore, the total number of perforations of the strength boundary N_C is

$$N_c = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_C |\mathbf{N}\cdot\dot{\mathbf{z}}|\, f(x, \alpha, y, \beta)dS\right\}d\alpha d\beta \tag{11}$$

If the random process $\mathbf{z}(t)$ is Gaussian, the probability density can be written as follows:

$$f(x_1, x_2, x_3, \ldots, x_n) = [2\pi]^{n/2}|M|^{-1/2}\exp\left\{\frac{-1}{2|M|}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}x_ix_j\right\} \tag{12}$$

where $|M|$ is the determinant of the time averages of the products and cross products of $x_1, x_2, x_3, \ldots, x_n$. M_{ij} is the cofactor of the ij element of $[M]$. If we let

$$x_1 = x(t) \qquad x_2 = \alpha(t) = \dot{x}(t)$$
$$x_3 = y(t) \qquad x_4 = \beta(t) = \dot{y}(t) \tag{13a}$$

then $|M|$ can be expressed as follows:

$$|M| = \left|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\begin{bmatrix} x(t)\\ \alpha(t)\\ y(t)\\ \beta(t)\end{bmatrix}\lfloor x(t)\,\alpha(t)\;\; y(t)\;\; \beta(t)\rfloor dt\right| \tag{13b}$$

or

$$[M] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\begin{bmatrix} x^2 & x\alpha & xy & x\beta\\ \alpha x & \alpha^2 & \alpha y & \alpha\beta\\ yx & y\alpha & y^2 & y\beta\\ \beta x & \beta\alpha & \beta y & \beta^2\end{bmatrix} dt \tag{13c}$$

The elements of $[M]$ can be determined by carrying out the integrations as indicated, or by power spectral analysis, where $\Phi_x(\omega)$ and $\Phi_y(\omega)$ are the power spectral density functions for $x(t)$ and $y(t)$, respectively. We obtain the following values for the ii elements:

$$\left.\begin{aligned}
\sigma_x{}^2 &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^2(t)dt = \int_0^\infty \phi_x(\omega)d\omega\\
\sigma_\alpha{}^2 &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \alpha^2(t)dt = \int_0^\infty \omega^2\Phi_x(\omega)d\omega\\
\sigma_y{}^2 &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} y^2(t)dt = \int_0^\infty \phi_y(\omega)d\omega\\
\sigma_\beta{}^2 &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \beta^2(t)dt = \int_0^\infty \omega^2\Phi_y(\omega)d\omega
\end{aligned}\right\} \tag{13d}$$

We obtain the following result for the (1, 2) and (2, 1) elements:

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\dot{x}(t)dt = 0 \tag{13e}$$

This result can be verified by integrating by parts and taking the limit.

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\dot{x}(t)dt = \lim_{T\to\infty}\frac{1}{2T}\times \left\{[x^2(t)]_{-T}^{T} - \int_{-T}^{T} x(t)\dot{x}(t)dt\right\} \tag{13f}$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\dot{x}(t)dt = \lim_{T\to\infty}\frac{1}{4T}[x^2(T) - x^2(-T)] = 0$$

This limit is zero since the respective values of $x^2(T)$ and $x^2(-T)$ are always finite, and they are divided by T which is arbitrarily large. In a similar manner, the (3, 4) and (4, 3) elements are zero.

Both the (1, 3) and (3, 1) elements are, by definition, equal to the cross correlation between the $x(t)$ and $y(t)$, and $\Phi_{xy}(\omega)$ is the cross power spectral density function for $x(t)$ and $y(t)$.

$$\rho\sigma_x\sigma_y = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)y(t)dt = \frac{1}{2}\int_{-\infty}^{\infty}\Phi_{xy}(\omega)d\omega \tag{13g}$$

The constant ρ is the correlation coefficient and can vary in the interval $(-1, 1)$.

Experience gained thus far in applying this analysis indicates that the other elements involving cross correlations between shear and axial stress quantities are negligibly small. They are assumed to be zero in the present analysis.

Therefore, the determinant M can be written as

$$|M| = \begin{vmatrix} \sigma_x{}^2 & 0 & \rho\sigma_x\sigma_y & 0\\ 0 & \sigma_\alpha{}^2 & 0 & 0\\ \rho\sigma_x\sigma_y & 0 & \sigma_y{}^2 & 0\\ 0 & 0 & 0 & \sigma_\beta{}^2\end{vmatrix} \tag{13h}$$

The frequency density can be expressed as follows:

$$f(x, \alpha, y, \beta) = \frac{1}{[2\pi]^2}\frac{1}{\sigma_x\sigma_\alpha\sigma_y\sigma_\beta(1-\rho^2)^{1/2}}\exp\left\{-\frac{1}{2}\left[\frac{\alpha^2}{\sigma_\alpha{}^2}+\frac{\beta^2}{\sigma_\beta{}^2}\right]\right\}\times \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_x{}^2}-\frac{2\rho xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y{}^2}\right]\right\} \tag{14}$$

If we assume that the variables x and y in Eq. (14) are the incremental or time variant quantities, varying from constant mean or steady flight values x_0 and y_0, respectively, then x and y can be replaced by $(x - x_0)$ and $(y - y_0)$.

After the mean values for x and y are inserted into Eq. (14), the number of crossings of a given curve C with velocities in the interval $(\alpha, \alpha + d\alpha;\ \beta, \beta + d\beta)$ is obtained from Eq. (10).

$$N_{c_{\alpha\beta}} = \frac{1}{[2\pi]^2}\frac{1}{\sigma_x\sigma_\alpha\sigma_y\sigma_\beta(1-\rho^2)^{1/2}}\times \exp\left\{\frac{-1}{2}\left[\frac{\alpha^2}{\sigma_\alpha{}^2}+\frac{\beta^2}{\sigma_\beta{}^2}\right]\right\}\int_C\left|\frac{\text{grad } C}{|\text{grad } C|}\cdot\dot{\mathbf{z}}\right|\times \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-x_0)^2}{\sigma_x{}^2}-\frac{2\rho(x-x_0)(y-y_0)}{\sigma_x\sigma_y}+\frac{(y-y_0)^2}{\sigma_y{}^2}\right]\right\}dSd\alpha d\beta \tag{15}$$

The total number of crossings or perforations of C/unit time is obtained by integrating Eq. (15) over all possible values of α and β.

$$N_c = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N_{c_{\alpha\beta}}\, d\alpha d\beta$$
$$= \frac{1}{[2\pi]^2}\frac{1}{\sigma_x\sigma_\alpha\sigma_y\sigma_\beta(1-\rho^2)^{1/2}}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\left[\frac{\alpha^2}{\sigma_\alpha{}^2}+\frac{\beta^2}{\sigma_\beta{}^2}\right]\right\}\left[\int_C\left|\frac{\text{grad } C}{|\text{grad } C|}\cdot\dot{\mathbf{z}}\right|\times\right.$$
$$\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-x_0)^2}{\sigma_x{}^2}\frac{-2\rho(x-x_0)(y-y_0)}{\sigma_x\sigma_y}+\right.\right.$$
$$\left.\left.\left.\frac{(y-y_0)^2}{\sigma_y{}^2}\right]\right\}dS\right]d\alpha d\beta \quad (16)$$

If we assume $x_0 = y_0 = 0$, we can obtain the familiar expression for the number of times/unit time that $\mathbf{z}(t)$ crosses constant levels of either $y = a$ or $x = b$, where a and b are arbitrarily assigned values.

The general equation for the number of crossings of any curve C per unit time [Eq. (16)] is used. Consider the number of crossings per unit time of the curve $y = a$.

The curve C is equal to

$$C = y - a = 0 \quad (17a)$$

and the gradient of C is

$$\text{grad}C = \frac{\partial C}{\partial x}\,\mathbf{i} + \frac{\partial C}{\partial y}\,\mathbf{j} = 0\mathbf{i} + \mathbf{j} \quad (17b)$$

and unit normal vector on C is

$$\mathbf{N} = \frac{\text{grad}C}{|\text{grad}C|} = 0\mathbf{i} + \mathbf{j} \quad (17c)$$

The absolute value of the velocity vector normal to C is

$$\left|\frac{\text{grad}C}{|\text{grad}C|}\cdot\dot{\mathbf{z}}\right| = |(0\mathbf{i}+\mathbf{j})(\alpha\mathbf{i}+\beta\mathbf{j})| = |\beta| \quad (17d)$$

If Eq. (16) is rewritten for this example, we obtain

$$N_c = \frac{1}{2\pi}\frac{1}{\sigma_\alpha\sigma_\beta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\beta|\exp\left\{-\frac{1}{2}\left[\frac{\alpha^2}{\sigma_\alpha{}^2}+\frac{\beta^2}{\sigma_\beta{}^2}\right]\right\}\times$$
$$d\alpha d\beta\int_{x=-\infty}^{x=\infty}\frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y(1-\rho^2)^{1/2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\times\right.$$
$$\left.\left[\frac{x^2}{\sigma_x{}^2}-\frac{2\rho xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y{}^2}\right]\right\}dx \quad (17e)$$

since

$$dS = \{1 + [(dy)/(dx)]^2\}^{1/2}\, dx = dx \quad (17f)$$

We note that the integral over x is identically equal to

$$\int_{x=-\infty}^{x=\infty} = \frac{1}{[2\pi]^{\frac{1}{2}}\sigma_y}\exp\left\{-\frac{1}{2}\frac{y^2}{\sigma_y{}^2}\right\} \quad (17g)$$

The double integral over α and β can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\beta|\exp\left\{-\frac{1}{2}\left[\frac{\alpha^2}{\sigma_\alpha{}^2}+\frac{\beta^2}{\sigma_\beta{}^2}\right]\right\}d\alpha d\beta =$$
$$\left[2\int_0^\infty \beta e^{-\beta^2/2\sigma_\beta{}^2}\,d\beta\right]\times$$
$$\left[[2\pi]^{1/2}\sigma_\alpha\int_{-\infty}^{\infty}\frac{1}{[2\pi]^{1/2}\sigma_\alpha}e^{-\alpha^2/2\sigma_\alpha{}^2}d\alpha\right] =$$
$$2\sigma_\beta{}^2[(2\pi)^{1/2}\sigma_\alpha] \quad (17h)$$

Substituting Eqs. (17g) and (17h) into Eq. (17e) gives

$$N_c = [1/(2\pi)][1/(\sigma_\alpha\sigma_\beta)]2\sigma_\beta{}^2[(2\pi)^{1/2}\sigma_\alpha]\times$$
$$\{1/[(2\pi)^{1/2}\sigma_y]\}e^{-y^2/2\sigma_y{}^2} = (1/\pi)(\sigma_\beta/\sigma_y)e^{-y^2/2\sigma_y{}^2} \quad (17i)$$

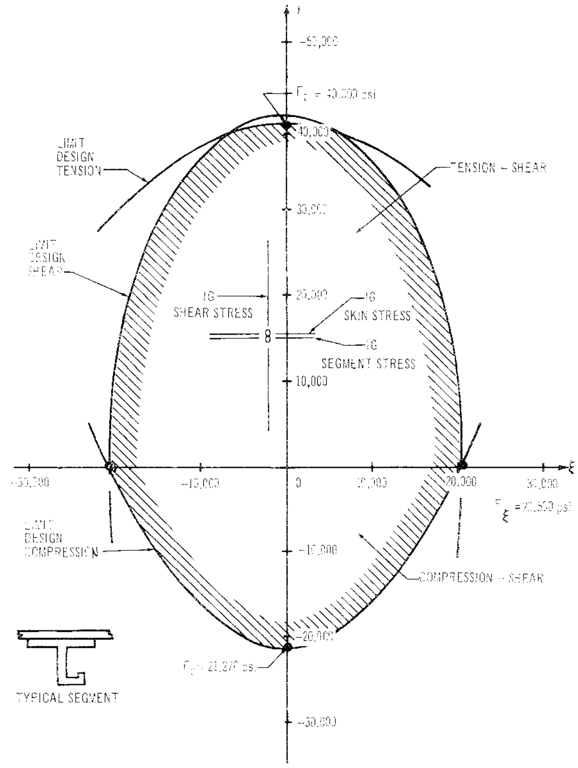

Fig. 4 Typical skin-stringer segment and strength interaction diagram.

This is the expression, first derived by Rice,[2] for the total number of crossings of a level $y = a$, where a is arbitrary in magnitude and in sign.

Perforations/Unit Time of an Arbitary Straight-Line Segment

The general expression for the number of crossings of curve C [Eq. (16)] is somewhat cumbersome to evaluate when

$$\left|\frac{\text{grad}C}{|\text{grad}C|}\cdot\dot{\mathbf{z}}\right| = g(x \text{ or } y,\ \alpha,\ \beta) \quad (18a)$$

However, a good approximate solution can be obtained if the curve C is considered to be a series of straight-line segments (Fig. 3). Then, for each segment k

$$\left|\frac{\text{grad}C_k}{|\text{grad}C_k|}\cdot\dot{\mathbf{z}}\right| = g_k(\alpha, \beta) \quad (18b)$$

The total number of crossings or perforations of the interaction envelope is

$$N_c = \sum_k |N_{ck}| \quad (19)$$

The number of crossings per unit time of a straight-line segment from (x_1, y_1) to (x_2, y_2) is derived as follows:

Consider a straight line through the points (x_1, y_1) and (x_2, y_2). The equation for this line is

$$y = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)(x - x_1) + y_1 \quad (20a)$$

or

$$y = m(x - x_1) + y_1 \quad (20b)$$

where m is the slope. Therefore, the curve C is

$$C = y - m(x - x_1) - y_1 = 0 \quad (20c)$$

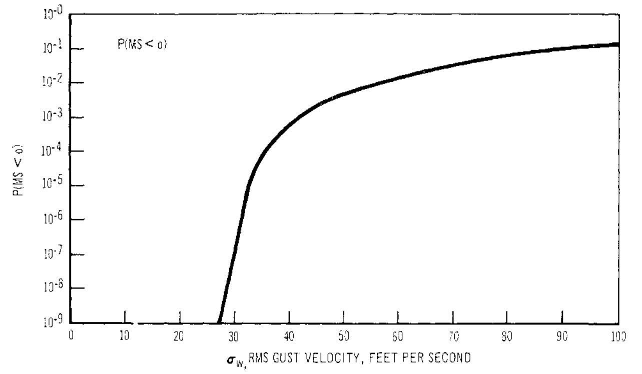

Fig. 5 Probability for exceeding limit design strength for various turbulence intensities.

and

$$\left|\frac{\text{grad}C}{|\text{grad}C|}\cdot\dot{\mathbf{z}}\right| = \left|\left(\frac{-m\mathbf{i}+\mathbf{j}}{[m^2+1]^{1/2}}\right)\cdot(\alpha\mathbf{i}+\beta\mathbf{j})\right|$$
$$|\mathbf{N}\cdot\dot{\mathbf{z}}| = \left|\frac{-m\alpha+\beta}{[m^2+1]^{1/2}}\right| \qquad (21)$$

Assuming $x_0 = y_0 = 0$ in Eq. (16), the number of perforations of the kth line segment approximating C is

$$N_{ck} = \frac{1}{[2\pi]^2}\frac{1}{\sigma_x\sigma_\alpha\sigma_y\sigma_\beta(1-\rho^2)^{1/2}} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{-m\alpha+\beta}{[m^2+1]^{1/2}}\right| \exp\left\{-\frac{1}{2}\left[\frac{\alpha^2}{\sigma_\alpha{}^2}+\frac{\beta^2}{\sigma_\beta{}^2}\right]\right\} d\alpha d\beta \times \int_{x_1,y_1}^{x_2,y_2}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_x{}^2}-\frac{2\rho xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y{}^2}\right]\right\} dS \qquad (22a)$$

Eq. (22a) can be rearranged as follows:

$$N_{ck} = \frac{1}{2\pi}\frac{1}{\sigma_\alpha\sigma_\beta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{-m\alpha+\beta}{[m^2+1]^{1/2}}\right| \times \exp\left\{-\frac{1}{2}\left[\frac{\alpha^2}{\sigma_\alpha{}^2}+\frac{\beta^2}{\sigma_\beta{}^2}\right]\right\} d\alpha d\beta\ \frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y(1-\rho^2)^{1/2}} \times \int_{x_1,y_1}^{x_2,y_2}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_x{}^2}-\frac{2\rho xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y{}^2}\right]\right\} dS \qquad (22b)$$

After integrating the double integral over α and β we obtain,

$$N_{ck} = \left(\frac{2}{\pi}\right)^{1/2}(m^2\sigma_\alpha{}^2+\sigma_\beta{}^2)^{1/2}\frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y(1-\rho^2)^{1/2}} \times \left(\frac{1}{m^2+1}\right)^{1/2}\int_{x_1,y_1}^{x_2,y_2}\exp\left\{\frac{-1}{2(1-\rho^2)} \times \left[\frac{x^2}{\sigma_x{}^2}-\frac{2\rho xy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y{}^2}\right]\right\} dS \qquad (22c)$$

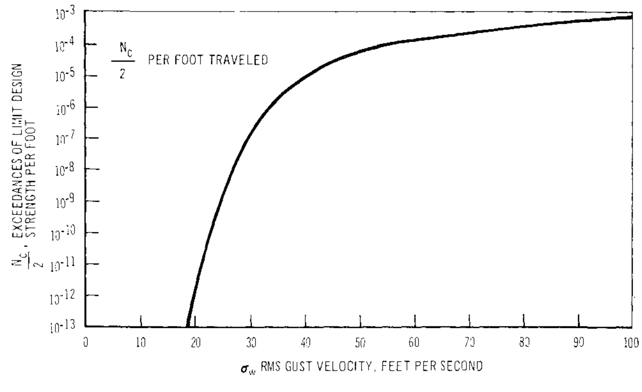

Fig. 6 Exceedances of limit design strength per foot traveled.

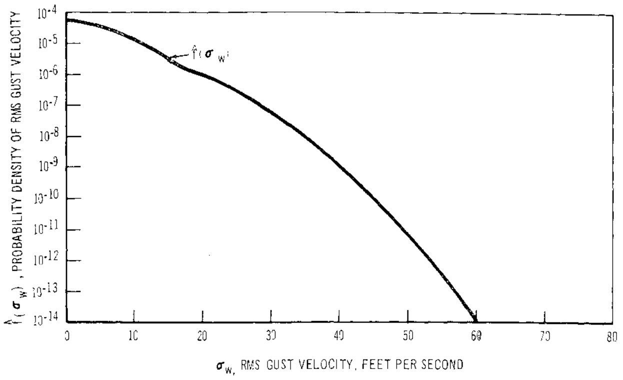

Fig. 7 Probability density of rms gust velocity.

Eq. (22c) shows that N_{ck} is the product of a slope-velocity function, which is constant for a given line segment and of a line integral over the joint probability function $p(x, y)$.

Numerical Example

Consider a stiffened skin-stringer segment, typical of those used in wing panel construction of subsonic jet transport airplanes. A cross section through the segment normal to the stringer is shown in Fig. 4. It is assumed that the airplane, of which this structural element is a part, is flying in turbulent air. It is of interest to determine (1) the probability of exceeding the limit design strength of the segment and (2) the expected number of times per hour that the limit strength would be exceeded for various root mean square (rms) gust velocities σ_w.

Numerical information pertinent to this example is given in Table 1. The appropriate limit strength interaction diagram is shown with the structural segment in Fig. 4. It will be noted that two level flight stress points, skin stress and segment stress, are shown in Table 1 and in Fig. 4. These are reference points for two joint probability stress distributions. The compression-shear segment stress probability distribution will be applied to the compression portion of the interaction diagram in Fig. 4, and the tension-shear or skin-stress distribution will be applied to the tension portion of the diagram.

The volume of the compression-shear joint probability density function within the compression region of the interaction diagram will be added to the volume of the tension-

Table 1 Numerical example

Dynamic gust analysis results	
Level flight stresses (based on appropriate effective areas)	
Skin stress, tension f_{0t}, psi	15,600
Segment stress, compression, f_{0c}, psi	15,200
Shear, ξ_0, psi	−2190
Dynamic analysis stress data (power spectral analysis—rms gust velocity $\sigma_w = 1.0$ fps)	
rms tension stress, $\bar{A}_{ft}$, psi	184.19
rms compressive stress, $\bar{A}_{fc}$, psi	179.51
rms shear stress, $\bar{A}_\xi$, psi	38.53
Correlation coefficient, ρ (Axial and shear stresses)	−0.2141
Zero crossings/sec	
Axial stress, N_{0f}	1.2205
Shear stress, $N_{0\xi}$	2.7235
rms stress rates, psi/sec	
Tension, $\sigma_{\beta t}$	1413
Compression, $\sigma_{\beta c}$	1377
Shear, σ_α	659
Allowable limit design stresses, psi	
Tension, F_t	40,000
Compression, F_c	21,270
Shear, F_S	20,500

shear density function in the tension region to determine the probability that the limit design strength will not be exceeded. The probability of exceeding the limit strength is determined by subtracting the summed probability from unity. The plot in Fig. 5 shows the probability of exceeding the limit design strength as a function of rms gust velocity σ_w. It has been assumed that all response quantities are proportional to the rms gust velocity. In order to determine the expected number of exceedances of the strength envelope per hour, it is necessary to determine the rms stress rates σ_α and σ_β. This information can be obtained from Rice's expression, [Eq. (17i)], the rms stresses, and the zero crossing rates.

Therefore

$$\sigma_{\beta_t} = 2\pi N_{0_t} \sigma_{f_t} \tag{23a}$$

$$\sigma_{\beta_c} = 2\pi N_{0_c} \sigma_{f_t} \tag{23b}$$

and

$$\sigma_\alpha = 2\pi N_{0_\xi} \sigma_\xi \tag{23c}$$

The rms stress rates for the example problem are given in Table 1. The expected number of exceedances of the strength envelope per ft traveled is a function of σ_w as shown in Fig. 6.

Probability density distributions of rms gust velocity for various altitudes have been estimated and reported in aeronautical literature.[3,4] A typical distribution of this type, $f(\sigma_w)$ vs σ_w, is shown in Fig. 7. The probability density distribution shown in Fig. 7 is applicable for the flight altitude of the example problem. Of course, the integral of $f(\sigma_w)$ over all σ_w is less than unity, because turbulent air is not present 100% of the time at any cruise altitude. The integral of $f(\sigma_w)$ is the fraction of time that rough air exists at the altitude in question.

The integral

$$\bar{G} = 1800\, V \int_0^\infty N_c f(\sigma_w) d\sigma_w \tag{24}$$

is an expression for the expected number of times per hour that the limit design strength will be exceeded. The expected number of exceedances of limit design strength per hour for the example problem is $4.178(10)^{-7}$. Figure 8 shows the distribution of limit design strength exceedances with rms gust intensity.

Conclusions

A procedure has been outlined for estimating the probability of exceeding the design strength of a structural element subjected to two random stress time-histories superimposed on steady stresses. The design strength envelope was assumed to be composed of stress interaction curves for axial and shear stresses. A probability calculation of this nature

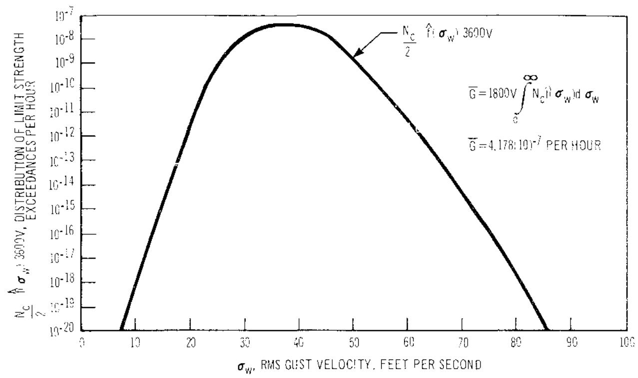

Fig. 8 Distribution of limit design strength exceedances with rms gust velocity.

[Eq. (4)] would indicate the fraction of time per unit time that the strength under combined stresses would be exceeded.

A second type, and perhaps a more meaningful statistical estimate, is the expected number of times per unit time that the combined random stresses would exceed the strength. The equations derived herein provide an estimate of the number of times per unit time that the strength curve is perforated or crossed. The assumption is that there is at least one peak value for the combined stresses for each pair of crossings. An expression for the number of crossings per unit time was derived for a general strength boundary curve C in Eq. (16) and for a strength boundary curve approximated by short straight-line segments [Eqs. (19) and (22c)].

The same general type of procedure could be applied to define strength margins for more than two random stress components if the strength interaction relationships were known. For example, if three random stress components were involved, the interaction or strength boundary would be a surface, and the number of perforations of this interaction surface per unit time could be determined.

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